Exercise 15

Prove the statement using the ε, δ definition of a limit and illustrate with a diagram like Figure 9.

$$\lim_{x\to 3}\left(1+\frac{1}{3}x\right)=2$$

Solution

According to Definition 2, proving this limit is logically equivalent to proving that

$$\text{if} \quad |x-3|<\delta \qquad \text{then} \qquad \left|\left(1+\frac{1}{3}x\right)-2\right|<\varepsilon$$

for all positive ε. Start by working backwards, looking for a number δ that's greater than $|x-3|$.

$$\left|\left(1+\frac{1}{3}x\right)-2\right|<\varepsilon$$

$$\left|\frac{1}{3}x-1\right|<\varepsilon$$

$$\left|\frac{1}{3}(x-3)\right|<\varepsilon$$

$$\frac{1}{3}|x-3|<\varepsilon$$

$$|x-3|<3\varepsilon$$

Choose $\delta=3\varepsilon$. Now, assuming that $|x-3|<\delta$,

$$\begin{aligned}\left|\left(1+\frac{1}{3}x\right)-2\right| &= \left|\frac{1}{3}x-1\right| \\ &= \left|\frac{1}{3}(x-3)\right| \\ &= \frac{1}{3}|x-3| \\ &< \frac{1}{3}\delta \\ &= \frac{1}{3}(3\varepsilon) \\ &= \varepsilon.\end{aligned}$$

Therefore, by the precise definition of a limit,

$$\lim_{x\to 3}\left(1+\frac{1}{3}x\right)=2.$$

Below is an illustration like Figure 9.